# Constructive analysis of noise-induced transitions for coexisting periodic attractors of the Lorenz model

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We study the stochastically forced Lorenz model in the parameter zone admitting two coexisting limit cycles under the transition to chaos via period-doubling bifurcations. Noise-induced transitions between both different parts of the single attractor and two coexisting separate attractors are demonstrated. The effects of structural stabilization and noise symmetrization are discussed. We suggest a stochastic sensitivity function technique for the analysis of noise-induced transitions between two coexisting limit cycles. This approach allows us to construct the dispersion ellipses of random trajectories for any Poincare sections. Possibilities of our descriptive-geometric method for a detailed analysis of noise-induced transitions between two periodic attractors of Lorenz model are demonstrated.

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## I. INTRODUCTION

Stochastic fluctuations of nonlinear oscillations play an important role for understanding the corresponding dynamical phenomena in electronic generators, lasers, mechanical, chemical, and biological systems. The various noise-induced transitions through periodic to more complicated regimes are a central problem in a nonlinear stochastic dynamics. The sensitivity analysis of random forced oscillations is a key for investigation of these transitions.

Analysis of nonlinear oscillations under stochastic disturbances was started by Pontryagin *et al.* [1] and continued by many researchers. One of the first fundamental monographs was noted in Ref. [2]. Effects of perturbations of limit cycles by noise were studied in [3,4]. Cycles of stochastic Brusselator and destruction of Hopf bifurcation by parametric noise were investigated in [5]. In [6,7] one can find an overview of approaches and results on stochastic oscillations of nonlinear systems.

Random disturbances for nonlinear dynamical systems can induce various phenomena such as stochastic resonance [8,9], noise-induced transitions [10], noise-induced order [11,12], noise-induced chaos [13], and noise-induced complexity [14]. An analysis of the effect of noise on dynamical systems with multiple stable states attracts the attention of many researchers [13,15–17]. Multistable systems exhibit complex dynamics [18] with noise-induced hopping between coexisting attractors and their basins of attraction [19–22]. A study of this basin-hopping phenomenon is important for an application to the problem of safety operation of systems with multistable stationary states under the random disturbances. Now a research interest is moved from a finding of qualitative variety for the noise-induced phenomena to a quantitative analysis of corresponding probabilistic mechanisms. Here the main problems are a spatial localization and a quantitative description of geometrical features of noise-induced stochastic attractors.

Noise-induced transitions between coexisting stable equilibria of one-dimensional (1D) and two-dimensional (2D) systems are well studied. Now a subject of intensive investigation is an analysis of noise-induced transitions between limit cycles of three-dimensional (3D) systems. Here a challenging problem is to study these transitions for parametrical zone close to chaos where high sensitivity of attractors generates noise-induced transitions even for small noise intensities.

Kolmogorov-Fokker-Planck (KFP) equation is a basic mathematical tool for the theoretical analysis of the noise-induced transitions. This equation gives the most detailed probabilistic description of the stochastic dynamics. However, a direct using of this equation is very difficult even for simplest situations. Under these circumstances, asymptotics and approximations can be used. For the approximation of KFP solutions a quasipotential technique is well known [23,24]. In the context of this technique, a stochastic sensitivity function (SSF) method for the probabilistic description of stochastic cycles was proposed [25–27]. In this paper, we apply this SSF method for a detailed geometrical description of noise-induced transitions between coexisting limit cycles of nonlinear stochastic 3D systems.

Since its invention [28] the Lorenz system has been a basic model for investigations in many directions. This 3D model shows a great variety of qualitatively different regimes of behavior [29]. The Lorenz model is a classic example of 3D system with coexisting periodic attractors and transition to chaos via period-doubling bifurcations. It allows us to use Lorenz model as a basic tool for the testing of new methods of nonlinear systems analysis.

Stochastically forced Lorenz model was investigated by direct simulation and analytically in [30–34]. Noise-induced oscillatory motions in the Lorenz equations were considered in [35]. Influence of noise on periodic attractors of the Lorenz model was studied in [36].

In this paper, we consider the stochastically forced Lorenz system with coexisting limit cycles for period-doubling bifurcation zone near chaos. The aim of our work is to analyze noise-induced transitions between periodic attractors. Our approach is based on the stochastic sensitivity function technique [25–27].

In Sec. II, we consider stochastically forced Lorenz model and study noise-induced phenomena for a period-doubling

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bifurcation zone. Noises deform deterministic unforced attractors and create corresponding stochastic attractors. Noise-induced transitions between both different parts of the single cycle and basins of attraction for coexisting separate cycles are demonstrated. The effect of a smoothing of the thin structure of deterministic attractor is shown in the stochastic bifurcation diagram. A structural stabilization and noise-induced symmetrization are discussed.

Method of constructive analysis of noise-induced transitions between two coexisting limit cycles is presented in Sec. III. SSF is a basic tool of our technique. This function provides a constructive approximation of stochastic 3D attractors. Theoretical background and algorithms of SSF technique are briefly described. This approach allows us to construct the dispersion ellipses of random trajectories for any Poincare sections. Possibilities of our approach for descriptive-geometric analysis of noise-induced transitions between coexisting limit cycles are demonstrated.

## II. NOISE-INDUCED PHENOMENA FOR PERIOD-DOUBLING BIFURCATION ZONE

### A. Attractors of the deterministic Lorenz model

Consider the deterministic Lorenz model,

$$\dot{x} = \sigma(-x+y) \quad \sigma = 10, \quad b = \frac{8}{3},$$

$$\dot{y} = rx - y - xz,$$

$$\dot{z} = -bz + xy, \tag{1}$$

on the interval $200 < r < 350$. This interval is well known [29] as a period-doubling bifurcation zone with infinite chain of limit cycles and transition to chaos. Here one can observe the following scenario of bifurcations as the parameter $r$ decreases from 350 to 200. Between $r=350$ and $r=313$ there is a single symmetric stable periodic orbit (limit cycle). Projection of limit cycle for system (1) with $r=330$ is presented in Fig. 1(a).

As the parameter $r$ decreases system (1) passes the symmetric saddle-node bifurcation and the symmetric stable cycle splits and two stable nonsymmetric cycles appear [see Fig. 1(b)]. Here we see a symmetry-breaking bifurcation. These two separate nonsymmetric cycles are observed between $r=313$ and $r=230$.

At the further decrease in $r$ system (1) passes a period-doubling bifurcation chain. Either of the two cycles demonstrates the standard sequence: cycle−2-cycle−4-cycle−… −$2^k$-cycle−$2^{k+1}$-cycle−… . In Figs. 1(c) and 1(d), one can see two separate 2-cycles and 4-cycles correspondingly. As the parameter $r$ decreases further these two regular periodic attractors move to two nonsymmetric chaotic ones. Then a distance between them vanishes, and one can observe a coalescing of these separate nonsymmetric chaotic attractors into a single symmetric chaotic attractor [see Fig. 1(e)]. So this scenario starts from a symmetry-breaking bifurcation and finishes by bifurcation of symmetrization.

A bifurcation diagram of the deterministic Lorenz model for $r \in [200,330]$ is presented in Fig. 2. Here $x$ coordinates of attractors intersection points with half-plane $y=0$ and $x \geq 0$ are plotted.

### B. Stochastic attractors of the forced Lorenz model

Consider a stochastically forced Lorenz system,

$$\dot{x} = \sigma(-x+y) + \varepsilon\dot{w}_1$$

$$\dot{y} = rx - y - xz + \varepsilon\dot{w}_2$$

$$\dot{z} = -bz + xy + \varepsilon\dot{w}_3. \tag{2}$$

Here $w_i(t)$ $(i=1,2,3)$ are independent standard Wiener processes with Gaussian increments, $E(w_i(t)-w_i(s))=0$ and $E(w_i(t)-w_i(s))^2=|t-s|$. Parameter $\varepsilon$ is a value of the noise intensity.

The noise disturbances result in a stochastic deformation of the deterministic unforced attractors. Under the random disturbances the trajectories of a stochastically forced system leave the deterministic attractor and form some bundle around it with a corresponding probabilistic distribution. A dispersion of random states of the stochastically forced system near a deterministic attractor depends on the noise intensity and stability properties of the attractor local parts.

In Fig. 3 stochastic attractors of system (2) for regular $(r=330, \; r=225)$ and chaotic $(r=200)$ zones and for noise intensity values $\varepsilon=1$, 5, and 10 are plotted. These intensity values allow us to visualize results of a direct simulation clearly.

For a numerical simulation of random trajectories of nonlinear stochastic systems various schemes can be used [37,38]. For Lorenz model considered here an appropriate stochastic component of random disturbances was introduced in the deterministic fourth-order Runge-Kutta scheme on each step $h=10^{-4}$.

To exclude the possibility that these results are just some peculiar effect of the particular step size and numerical scheme used, the calculations were repeated with other step sizes and schemes to see if similar behavior is obtained.

For small noises, random states are concentrated close to deterministic cycle [see Fig. 3(a)]. As noise intensity grows, the dispersion of random trajectories increases [see Fig. 3(b)]. With further increase in the noise, the transitions between different parts of the attractor are observed [see transition trajectories at the center of the attractor in Fig. 3(c)]. The probability of this type of noise-induced transitions depends on both the stochastic sensitivity and a spatial arrangement of the different attractor parts.

Consider the case of a noise-induced transition between two separate coexisting deterministic cycles. For small noises, random states are concentrated in small neighborhoods of corresponding deterministic curves. As noise intensity grows, the dispersion increases, two separate bundles of random trajectories approach and get mixed up. Probabilistic characteristics of these noise-induced transitions between basins of attraction for coexisting unforced cycles depend on both the stochastic sensitivity and a spatial arrangement of these deterministic attractors.

For any nonsingular noises $(\varepsilon > 0)$ random trajectories with nonzero probability cross separatrix between basins of

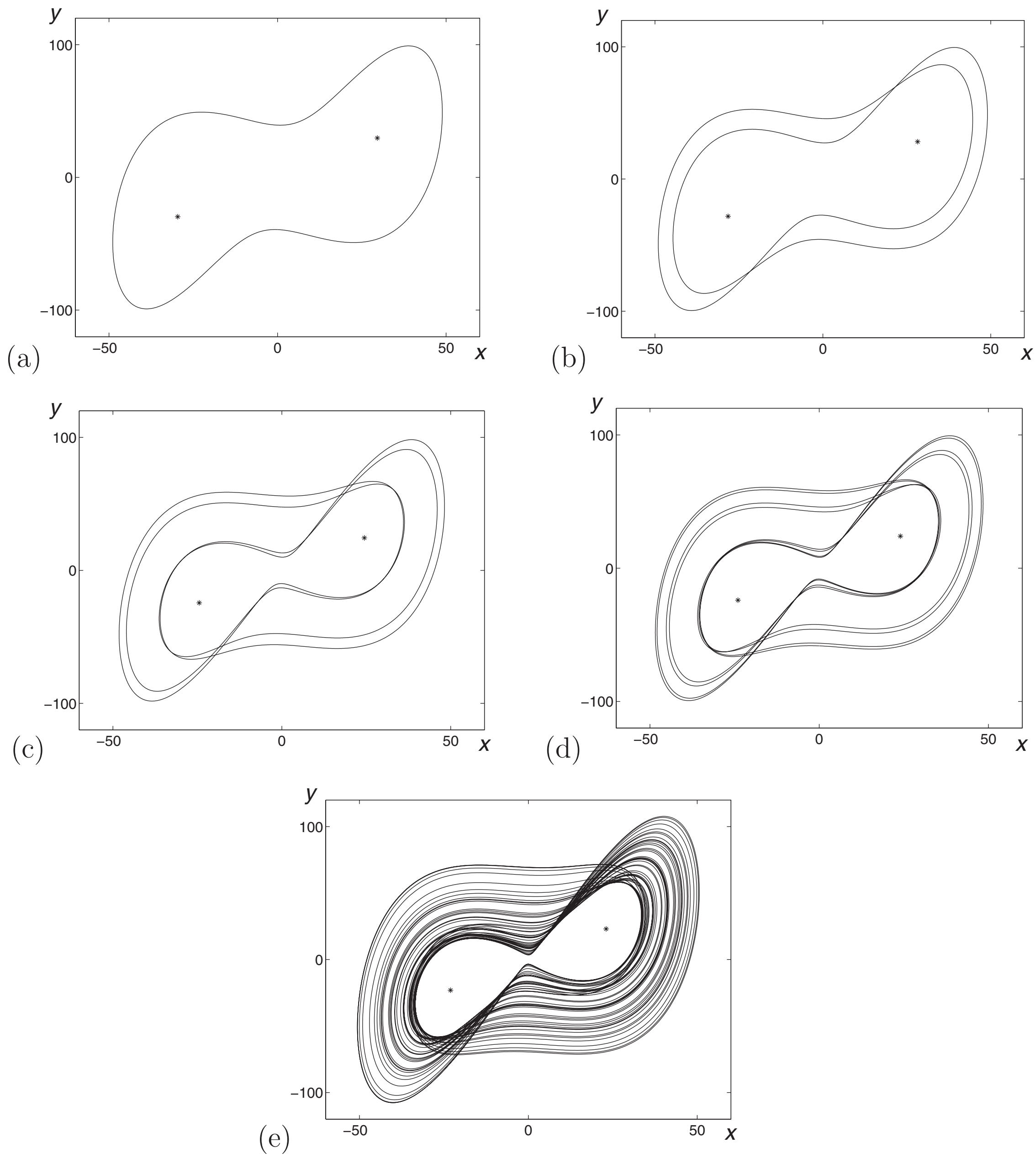


FIG. 1. Attractors of the deterministic Lorenz model: (a) $r=330$; (b) $r=300$; (c) $r=225$; (d) $r=217$; and (e) $r=200$.

attraction of two separate coexisting limit cycles. So, under the random disturbances two separate deterministic attractors are combined in one united stochastic attractor. For example, system (2) noises transform two nonsymmetric limit cycles into one symmetric stochastic attractor. In Figs. 3(d)–3(f) we observe this noise-induced symmetrization. Influence of noises results in a smoothing of the thin structure of deterministic attractors. A stochastic bifurcation diagram clearly reflects this effect even for small noises (see Fig. 4).

In Figs. 3(g)–3(i), for fixed $r=200$ and various values of noise intensity $\varepsilon=1$, 5, and 10 stochastically forced chaotic attractors of system (1) are plotted. As we can see, stochastic attractors in Figs. 3(g)–3(i) are similar. Moreover, these attractors are close to chaotic attractor of unperturbed Lorenz system (1) [see Fig. 1(e)]. So, presence of noises results in structural stabilization of nonlinear system dynamics for a wide range of parameters.

The results presented here give us a qualitative description of possible noise-induced transitions and corresponding phenomena. More detailed quantitative analysis of noise-induced transitions between two coexisting cycles will be presented in Sec. III using the SSF technique.

## III. ANALYSIS OF THE STOCHASTIC ATTRACTORS AND NOISE-INDUCED TRANSITIONS

### A. SSF technique

Consider the deterministic system of nonlinear differential equations,

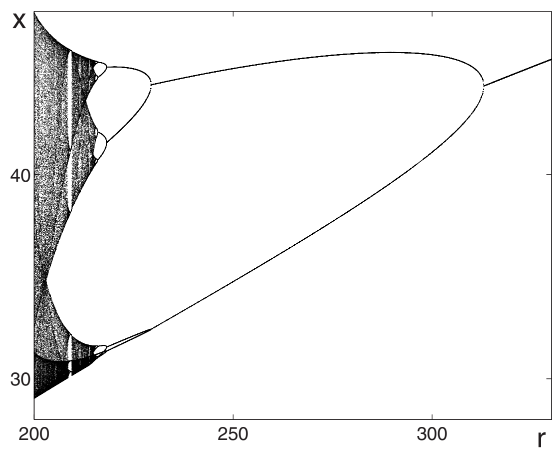


FIG. 2. Bifurcation diagram of the deterministic Lorenz model.

$$\dot{x} = f(x). \tag{3}$$

Suppose system (3) has a $T$-periodic solution $x=\xi(t)$ with an exponentially stable phase curve $\gamma$. It means that for a small neighborhood $\Gamma$ of the cycle $\gamma$ there exist constants $K>0$, $l>0$ such that for any solution $x(t)$ of system (3) with $x(0)=x_0 \in \Gamma$ the following inequality holds

$$\|\Delta[x(t)]\| \leq Ke^{-lt}\|\Delta(x_0)\|.$$

Here $\Delta(x)=x-\gamma(x)$ is a deviation of a point $x$ from a cycle $\gamma$ and $\gamma(x)$ is a point on cycle $\gamma$ that is nearest to $x$.

Systems of stochastic differential equations (in Ito's or Stratonovich's sense) are traditional mathematical models allowing us to study a quantitative description of results of external disturbances. In this paper, along with the unforced deterministic system (3) we consider Ito's stochastic system

$$\dot{x} = f(x) + \varepsilon\sigma(x)\dot{w}. \tag{4}$$

Here $w(t)$ is a $n$-dimensional Wiener process, $\sigma(x)$ is $n\times n$-matrix function of disturbances with intensity $\varepsilon$.

The random trajectories of forced system (4) leave the closed curve of deterministic cycle $\gamma$ and due to cycle stability form some bundle around it.

The detailed description of random distribution dynamics of this bundle is given by KFP equation. If the character of a transient is inessential and the main interest is connected with regime of steady-state stochastic auto-oscillations, then it is possible to restrict the research by analysis of a station-

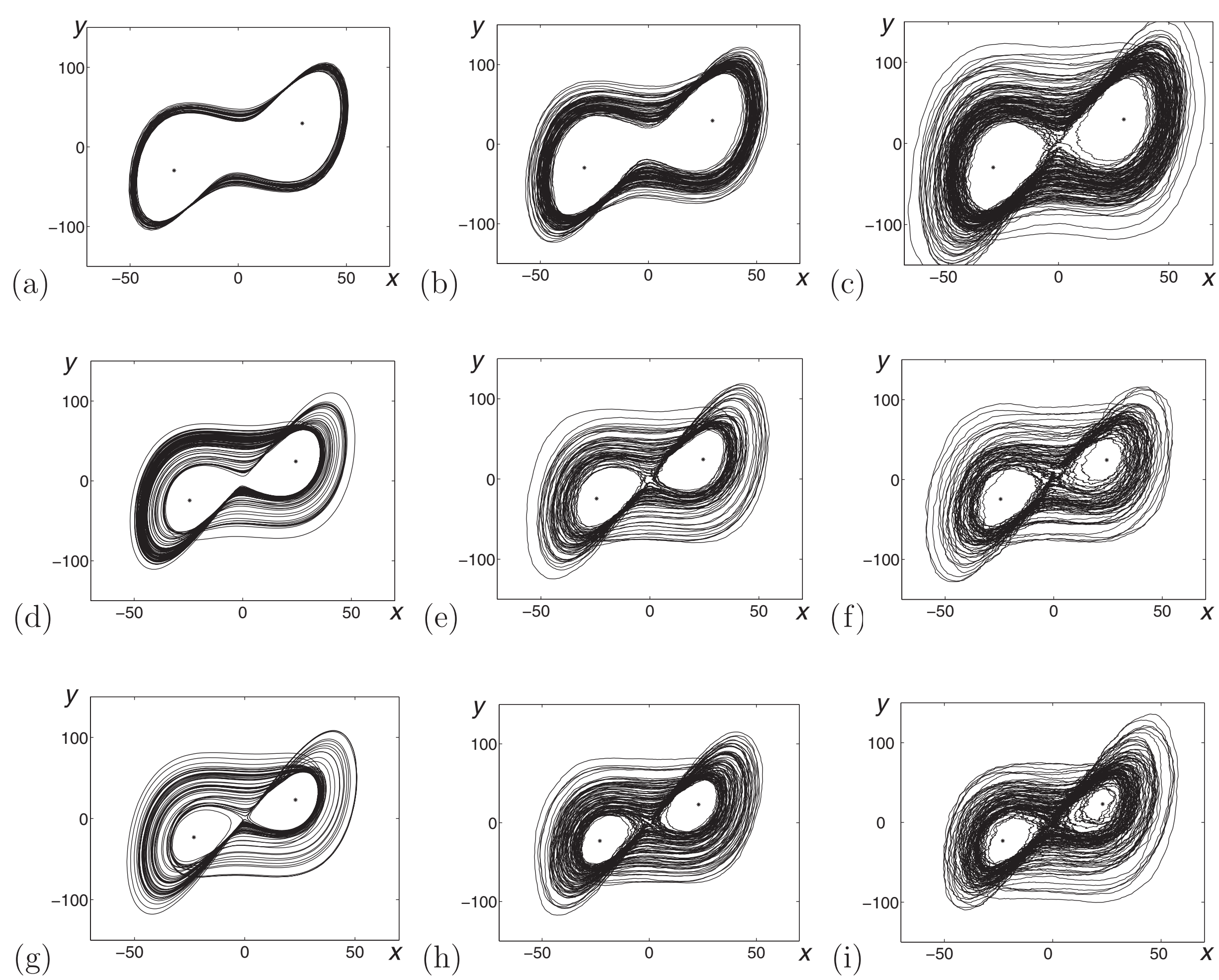


FIG. 3. Stochastic attractors of the forced Lorenz model: (a) $r=330$ and $\varepsilon=1$; (b) $r=330$ and $\varepsilon=5$; (c) $r=330$ and $\varepsilon=10$; (d) $r=225$ and $\varepsilon=1$; (e) $r=225$ and $\varepsilon=5$; (f) $r=225$ and $\varepsilon=10$; (g) $r=200$ and $\varepsilon=1$; (h) $r=200$ and $\varepsilon=5$; (i) $r=200$ and $\varepsilon=10$.

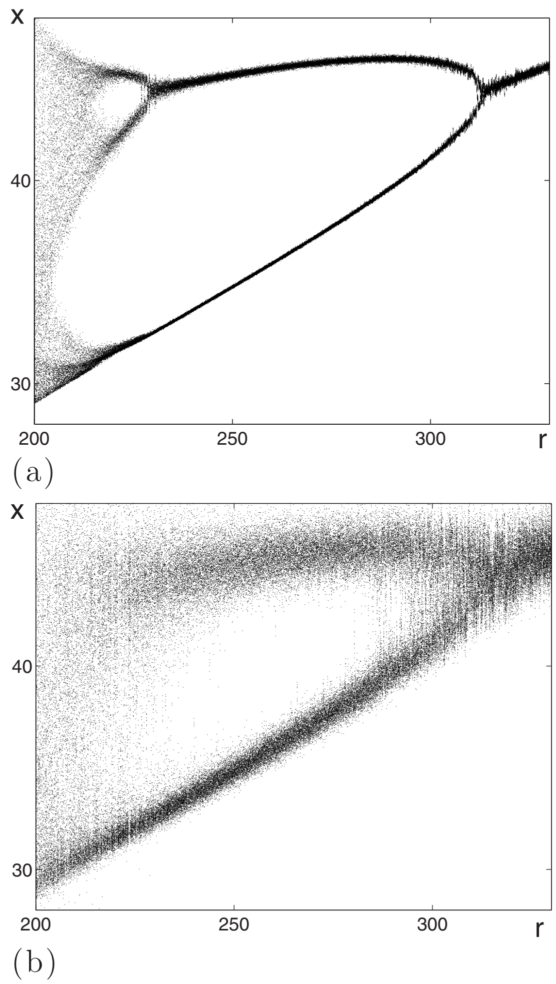


FIG. 4. Bifurcation diagram for the stochastically forced Lorenz model: (a) $\varepsilon=0.1$ and (b) $\varepsilon=1$.

ary density function $\rho(x,\varepsilon)$. This stationary probabilistic distribution defines a corresponding stochastic attractor. Analytical research of the stationary KFP equation for stochastic limit cycles considered here is a very difficult problem. Under these circumstances asymptotics and approximations based on quasipotential $v(x)=-\lim_{\varepsilon\to 0}\varepsilon^2\log\rho(x,\varepsilon)$ are actively used [6,23,24].

The probabilistic distribution for the bundle of random trajectories localized near cycle has Gaussian approximation [25,27]

$$\rho\approx Ke^{-v(x)/\varepsilon^2}\approx K\exp\left\{-\frac{[\Delta(x),\Phi^{+}(\gamma(x))\Delta(x)]}{2\varepsilon^2}\right\},$$

with covariance matrix $\varepsilon^2\Phi(\gamma)$. Here "+" means a pseudoinversion. The covariance matrix characterizes the dispersion of the points of intersection of random trajectories with hyperplane orthogonal to cycle at the point $\gamma$. The function $\Phi(\gamma)$ is a SSF of the limit cycle. This function allows us to describe nonuniformity of a bundle width along cycle for all directions. It gives a simple way to indicate the most and the least sensitive parts of the cycle under the influence of external noises.

It is convenient to search for a function $\Phi(\gamma)$ in a parametric form. The solution $\xi(t)$ connecting the points of cycle $\gamma$ with points of an interval $[0,T)$ gives the natural parametrization $\Phi[\xi(t)]=W(t)$. A matrix function $W(t)$ is a solution of the boundary value problem for Lyapunov equation,

$$\dot{W}=F(t)W+WF^{\top}(t)+P(t)S(t)P(t), \tag{5}$$

with conditions

$$W(t)r(t)\equiv 0, \tag{6}$$

$$W(0)=W(T). \tag{7}$$

Here

$$F(t)=\frac{\partial f}{\partial x}[\xi(t)],\quad S(t)=\sigma[\xi(t)]\sigma^{\top}[\xi(t)],$$

$$r(t)=f[\xi(t)],\quad P(t)=P_{r(t)},\quad P_r=I-rr^{\top}/r^{\top}r,$$

where $P_r$ is a projection matrix onto the subspace orthogonal to the vector $r\neq 0$. Details and mathematical background can be found in [27]. Condition (6) means that the matrix $W(t)$ is singular [$\det W(t)\equiv 0$].

For 3D cycles, due to singularity, the matrix $W(t)$ has the following decomposition:

$$W(t)=\lambda_1(t)v_1(t)v_1^{\top}(t)+\lambda_2(t)v_2(t)v_2^{\top}(t). \tag{8}$$

Here $\lambda_1(t)\geq\lambda_2(t)\geq\lambda_3(t)\equiv 0$ are eigenvalues and $v_1(t),v_2(t),v_3(t)$ are eigenvectors of the matrix $W(t)$.

The constructive method for computation of this decomposition is presented in [27]. Denoted by $u_1(t)$ and $u_2(t)$ some orthonormal basis of the plane are orthogonal to the cycle at the point $\xi(t)$. The eigenvectors $v_1(t)$ and $v_2(t)$ can be represented by rotation of bases $u_1(t)$ and $u_2(t)$ with some angle $\varphi(t)$

$$v_1(t)=u_1(t)\cos\varphi(t)+u_2(t)\sin\varphi(t),$$

$$v_2(t)=-u_1(t)\sin\varphi(t)+u_2(t)\cos\varphi(t). \tag{9}$$

Thus, the decomposition [Eqs. (8) and (9)] allows us to express an unknown solution of a system [Eqs. (5)–(7)] by means of three scalar functions $\lambda_1(t)$, $\lambda_2(t)$, and $\varphi(t)$.

**Theorem 1.** The matrix $V(t)$ is the solution of a system [Eqs. (5) and (6)] if and only if scalar functions $\lambda_1(t)$, $\lambda_2(t)$, and $\varphi(t)$ of decompositions (8) and (9) satisfy a system

$$\dot{\lambda}_1=\lambda_1v_1^{\top}[F+F^{\top}]v_1+v_1^{\top}Sv_1,$$

$$\dot{\lambda}_2=\lambda_2v_2^{\top}[F+F^{\top}]v_2+v_2^{\top}Sv_2,$$

$$(\lambda_1-\lambda_2)\dot{\varphi}=\lambda_2v_1^{\top}Fv_2+\lambda_1v_1^{\top}F^{\top}v_2+v_1^{\top}Sv_2-(\lambda_1-\lambda_2)\dot{u}_1^{\top}u_2. \tag{10}$$

The matrix $W(t)$ (required stochastic sensitivity function of cycle) of system [Eqs. (5)–(7)] solution can be obtained by the following limit procedure [27].

**Theorem 2.** Let $T$-periodic matrix $W(t)$ be a solution of systems (5)–(7).

Let $\lambda_1(t)$, $\lambda_2(t)$, and $\varphi(t)$ be an arbitrary solution of system (10) on the interval $[0,+\infty)$. Put $V(t)=\lambda_1(t)P_1(t)+\lambda_2(t)P_2(t)$, where $P_i(t)=v_i(t)v_i^\top(t)$ with vector functions $v_i(t)$ obtained from Eq. (9). Then matrix $V(t)$ tends to matrix $W(t)$ as $t\to+\infty$;

$$\lim_{t\to+\infty}[V(t)-W(t)]=0.$$

The matrix $W(t)$ is determined by scalar functions $\lambda_1(t),\lambda_2(t)$ and vectors $v_1(t)$ and $v_2(t)$. In a case of nondegenerate noises the functions $\lambda_1(t),\lambda_2(t)$ are strictly positive and determine for any $t$ a dispersion of random trajectories around cycle along vectors $v_1(t),v_2(t)$.

Values $\lambda_1(t)$ and $\lambda_2(t)$ determine the size and $v_1(t),v_2(t)$ determine the directions of a dispersion ellipse axes. The equation of this ellipse in a plane orthogonal to the cycle $\gamma$ at the point $\xi(t)$ looks similar to

$$[x-\xi(t)]^\top W^+(t)[x-\xi(t)]=2k^2\varepsilon^2,$$

where the parameter $k$ determines a fiducial probability $p=1-e^{-k}$.

The possibility of SSF to predict some peculiarities of dynamic for stochastically and periodically forced Brusselator is presented in [25] where critical values of Brusselator parameters have been found. For these values of parameters Brusselator has a high sensitive dependence to the noise without sensitive dependence to initial conditions. A very small (in fact background) noise transfers a system to a chaotic regime. As it was demonstrated in [26,39], SSF technique is a useful constructive tool for investigation of stochastic cycles of 3D systems in a period-doubling bifurcations zone. In these papers, it was shown that transition across period-doubling bifurcations zone from order to chaos is accompanied by an essential growth of the stochastic sensitivity.

### B. Analysis of noise-induced transitions for Lorenz model

In this section, we apply SSF technique to the analysis of noise-induced transitions between stochastic cycles of the forced Lorenz model [Eq. (2)].

The eigenvalues $\lambda_1(t)>\lambda_2(t)$ of the stochastic sensitivity matrix $W(t)$ are convenient scalar characteristics of the cycle sensitivity. For the stochastic cycle of Lorenz model with $r=330$ the plots of functions $\lambda_1(t),\lambda_2(t)$ (solid lines) are shown in Fig. 5. Results of the direct numerical simulation for $\varepsilon=0.01$ are marked in Fig. 5 by asterisks. One can see that theoretical curves $\lambda_1(t),\lambda_2(t)$ are arranged near the values of empirical sensitivity function and clearly reflect the main features (sharp peaks and monotonicity intervals) of this function. Figure 5 shows peculiarities of random bundle behavior such that nonuniformity of bundle width along the cycle and large dispersion overfall in normal directions. Plots of functions $\lambda_1(t),\lambda_2(t)$ for stochastic 2-cycle of system (2) for $r=225$ are shown in Fig. 6. As one can see, 2-cycle is more stochastically sensitive.

A relation between behavior of eigenvalues $\lambda_1$ and $\lambda_2$ in time and spatial peculiarities of the random trajectories for the stochastic cycle of Lorenz model can be illustrated at

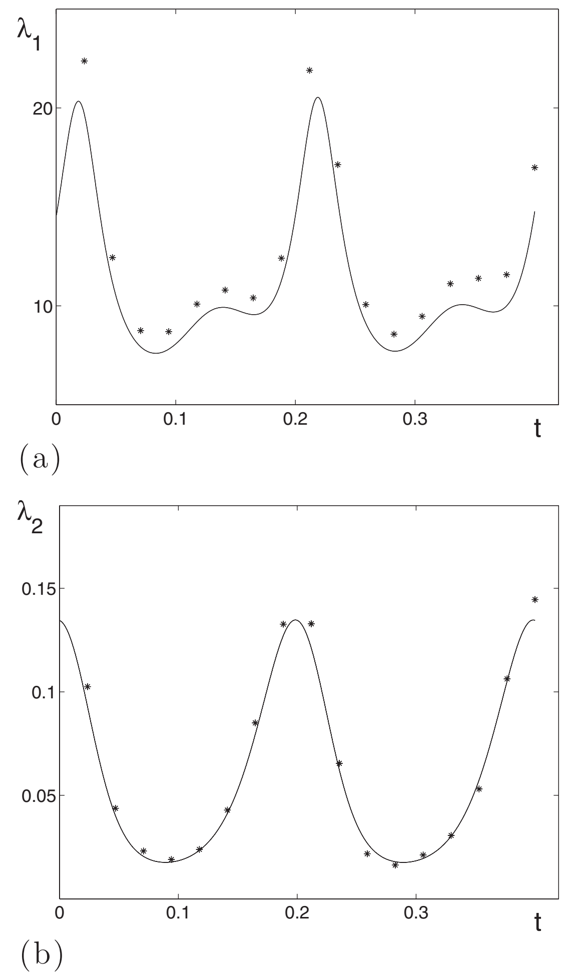


FIG. 5. Stochastic sensitivity of the cycle for $r=330$ and $\varepsilon=0.01$.

comparison of Figs. 3(a) and 5. Indeed, extremes of the functions $\lambda_1$ and $\lambda_2$ indicate corresponding spatial zones of the stochastic cycle where random trajectories have maxima and minima of their dispersions.

Our approach to the analysis of noise-induced transitions between attractors of the stochastically forced system is based on the quantitative study of the random states dispersions around deterministic attractors. For small noises, random states are concentrated near deterministic attractors. As noise intensity grows, the dispersion of random states increases and the transitions between separate attractors are observed. For the detailed geometrical investigation of noise-induced transitions in 3D case, we use Poincare section technique.

The functions $\lambda_1(t)$, $\lambda_2(t)$, $v_1(t)$, and $v_2(t)$ allow us to construct an ellipse of the dispersion of random trajectories intersection points with any Poincare section plane. As one can see in Fig. 7 the constructed ellipse precisely reflects a spatial dispersion of the intersection points. For the ellipse plotted in Fig. 7 the value of fiducial probability $p$ equals 0.85.

So, the SSF technique allows us to construct the dispersion ellipses for any Poincare sections. These ellipses give us a descriptive-geometric method for the analysis of noise-

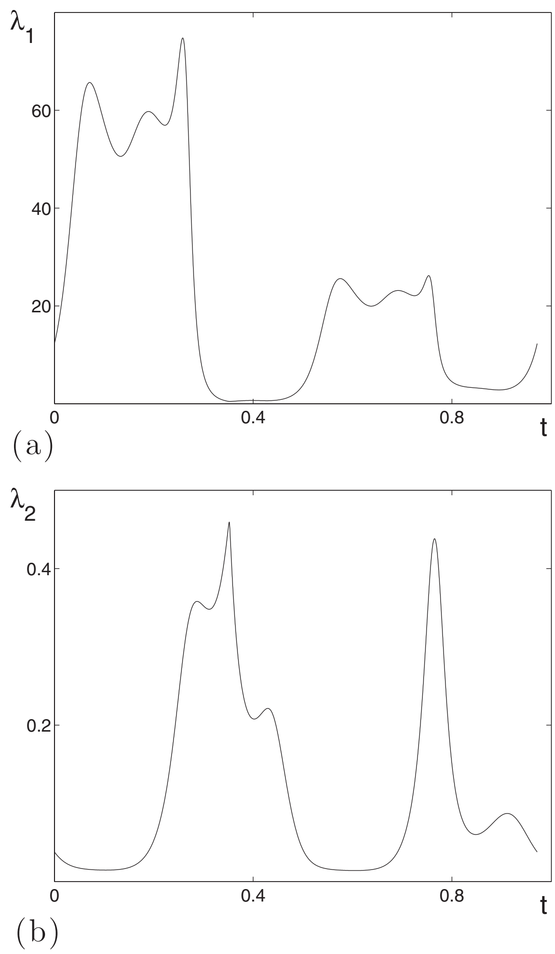


FIG. 6. Stochastic sensitivity of 2-cycle for $r=225$.

induced transitions between coexisting limit cycles.

Compare results of SSF technique with direct numerical simulation. Let us track sequential phases of noise-induced transitions between two separate coexisting 3D cycles with the help of Poincare sections. In Fig. 8, corresponding dispersion ellipses (solid line) found by SSF technique and intersection points (asterisks) of the random trajectories with a half plane $y=0$ and $x>0$ for fixed $r=300$ and various values of noise intensity $\varepsilon=0.3$, 1, and 2 are plotted.

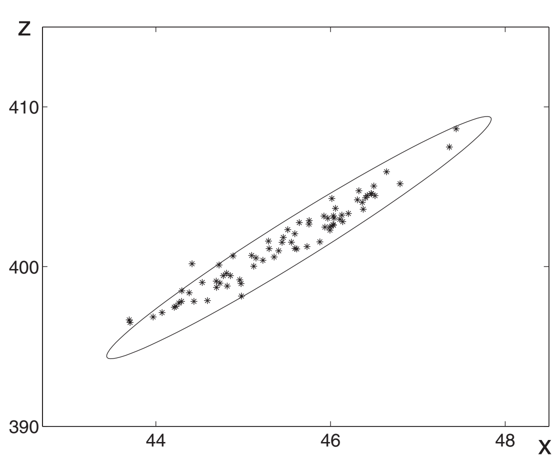


FIG. 7. Poincare section of the stochastic cycle and confidence ellipse for $r=330$, $\varepsilon=1$, and $p=0.85$.

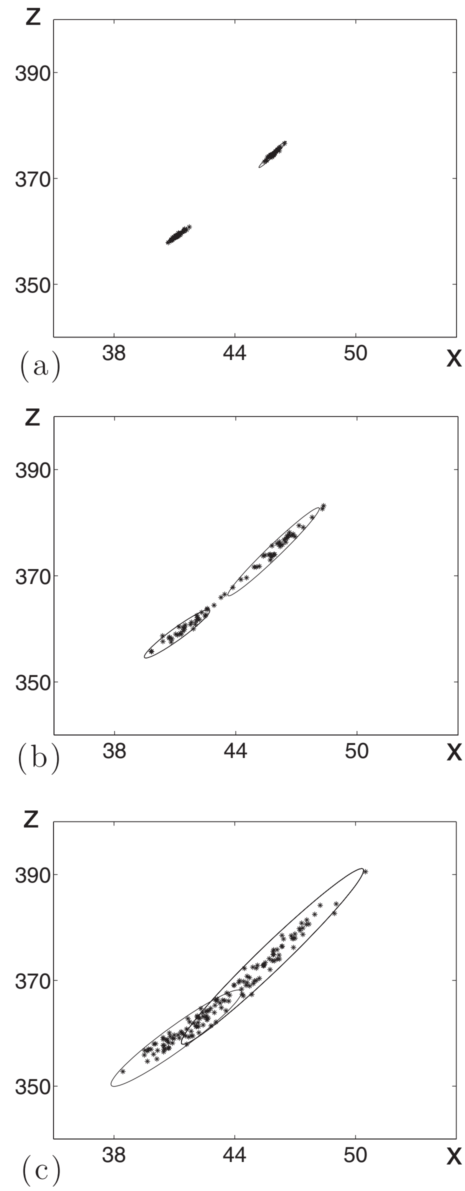


FIG. 8. Noise-induced transitions $r=300$: (a) $\varepsilon=0.3$; (b) $\varepsilon=1$; (c) $\varepsilon=2$.

For $\varepsilon=0.3$, random trajectories are localized near deterministic orbits [see Fig. 8(a)]. For $\varepsilon=1$, intersection points approach [see Fig. 8(b)] and for $\varepsilon=2$ become essentially intermixed [see Fig. 8(c)].

These ellipses clearly reflect essential peculiarities of random states distribution near the deterministic cycle. For $\varepsilon=0.3$, dispersion ellipses are localized near deterministic orbits [see Fig. 8(a)] and lie far from one another. As parameter $\varepsilon$ grows, these ellipses approach and begin to intersect [see Figs. 8(b) and 8(c)]. This intersection gives indication of the noise-induced transition beginning. Our ellipses technique based on SSF method connects a level of noise-induced transition with a noise intensity parameter. In fact, a size and a mutual arrangement of dispersion ellipses allow us to describe and predict effectively the main features of noise-induced transitions.

## IV. CONCLUSION AND DISCUSSION

We studied a basin-hopping phenomenon for systems with multistable states under the random disturbances. This paper concentrated on the noise-induced transitions in 3D systems with limit cycles on the period-doubling route to chaos. The main probabilistic phenomena and methods of their analysis are presented for the well-known stochastically forced Lorenz model. We studied noise-induced transitions between both different parts of the single forced limit cycle and two coexisting separate cycles. Our analysis based on the direct numerical simulation showed that the random disturbances generate the effects of the structural stabilization and noise symmetrization of stochastic attractors of the Lorenz system.

In this paper, we proposed a universal theoretical approach to the quantitative and geometrical analysis of the probabilistic mechanism of noise-induced transitions between coexisting 3D-limit cycles. This approach is based on Poincare sections method and stochastic sensitivity function technique. This function [25–27] provides a constructive approximation of the probabilistic distribution for stochastic 3D cycles. In the presented paper it was shown that SSF technique is an effective method for noise-induced transitions analysis. This technique allows us to find a spatial configuration and sizes of the dispersion ellipses of random trajectories for any Poincare sections. Dispersion ellipses are a plain and useful tool for the study of the probabilistic mechanism of noise-induced phenomena. For small noise, the dispersion ellipses are localized near deterministic cycles and well separated. As noise intensity grows, these ellipses approach one to another and begin to intersect. This intersection indicates that noise-induced transition begins. In fact, a size and a spatial arrangement of dispersion ellipses allow us to describe and predict effectively the main features of the noise-induced transitions without huge costs for direct numerical simulations of random trajectories.

## ACKNOWLEDGMENTS

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